Int. J. Solids Structures, 1975, Vol. 11, pp. 725–740. Pergamon Press. Printed in Great Britain

UNIAXIAL WAVE PROPAGATION IN A NONLINEAR THERMOVISCOELASTIC MEDIUM WITH TEMPERATURE DEPENDENT PROPERTIES

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(Received 8 February 1974; revised 7 November 1974)

Abstract—The problem of finite wave propagation in a nonlinearly thermoviscoelastic thin rod whose viscoelastic properties are temperature dependent is considered. The rod is subjected to mechanical or thermal time-dependent loading. The coupled equations of motion and heat conduction are based on a constitutive theory of nonisothermal nonlinear viscoelasticity which is described by single-integral terms only. This theory is reformulated here for the uniaxial motion of a compressible rubbery material. The solution of the field equations is obtained by a numerical procedure which is developed for the present case and is able to handle successfully shock waves whenever they built up in the nonlinear material.

INTRODUCTION

Nonlinear viscoelastic constitutive equations can be described by a polynomial expansion of a multiple integral expression, see for example Lockett [1] and references cited there. Experimental determination of the material functions involved in this description requires a large number of tests which would be prohibitive and impractical especially for significant deviations from infinitesimal deformations. For this reason approximate single-integral constitutive relations which incorporate the nonlinear behavior were considered by several investigators [1].

Temperature dependence of nonlinear viscoelastic behavior which is described in terms of general constitutive functionals was incorporated by Lianis [2] by generalizing the concept of thermorheologically simple behavior to a nonlinear one. Later [3] this was applied to describe the nonlinear nonisothermal behavior of an incompressible isotropic material. This description is based on the concept of finite linear viscoelastic theory which contains single-integrals only. Due to the incompressibility, the material functions in the nonisothermal constitutive relations in [3] differ from the isothermal relations only in that they are evaluated at the "reduced time".

Another theory which is applied to nonlinear thermoviscoelastic behavior and contains only single-integral terms was developed by Schapery [4]. This theory is based on the thermodynamics of irreversible processes and the single-integral constitutive equations have a form which is very similar to the Boltzmann superposition integrals of the linear theory of viscoelasticity. Later [5] this theory was shown to be consistent with the mechanical behavior reported for several metals and polymeric solids. Also, it was applied to characterize several nonlinear viscoelastic materials [6] and a nonlinear fiber-reinforced composite [7].

In this paper we employ Schapery theory in order to solve the problem of uniaxial finite amplitude wave propagation in a nonlinear thermoviscoelastic compressible material. We start by evaluating a nonlinear thermoelastic uniaxial constitutive relation for a compressible material. This relation is then employed in order to construct a constitutive equation, for a nonlinear thermoviscoelastic compressible material, which contains single-integral terms according to Schapery theory for rubbery polymers which can sustain very high deformations before breaking. The time-dependent temperature field is governed by a coupled energy equation together with the associated dissipation function. The energy equation and the dissipation function are expressed in

[4] in terms of observed and hidden coordinates. For the present uniaxial case they are transformed and given here in terms of the observed coordinates and temperature only. By adopting the Fourier law for the heat flux, we are able to formulate the coupled nonlinear equation of motion and heat conduction equation for the present viscoelastic compressible material, and each equation contains single-integral terms only. The viscoelastic modulus is described here by a polynome of exponential series which in the linear range yields the generalized Maxwell model.

In the linear range it is shown that the constitutive relation, heat conduction equation and the associated dissipation function reduce to those given by Hunter[8] for a thermorheologically simple solid.

A finite-difference numerical scheme is formulated to the present system of nonlinear coupled integrodifferential equations. When, due to nonlinearities, a discontinuous solution is developed (shock wave), an iterative procedure is applied in order to remove the numerical oscillations known to be present near shocks in numerical solutions. This iterative method was previously applied by the authors in studying nonlinear elastic wave propagation in elastic[9] and thermoelastic[10] media and is generalized here and applied to the present nonlinear thermoviscoelastic wave propagation. In both the finite elastic and thermoelastic cases, which are obtained as special cases of the present formulation, the proposed numerical scheme reduces basically to those investigated in [9] and [10] and whose accuracy was checked by comparison with several situations for which analytical conclusions could be derived.

Results are given for both mechanical and thermal loading of a nonlinear semi-infinite thermoviscoelastic rod with temperature dependent properties. Other cases of nonlinearly elastic, viscoelastic, thermoelastic and thermoviscoelastic with temperature independent properties rod are shown for comparison. Results are also shown for a linear thermoviscoelastic rod with temperature dependent properties, showing clearly the effect of nonlinearity. Even in this "linear" case there are two sources of nonlinearity due to the dependence of the material properties on the unknown temperature, and the appearance of the quadratic dissipation function.

Distortions of the simple wave solution obtained from a smoothly decreasing input in the nonlinear elastic case, due to the various mechanisms involved are shown. Also, the behavior of a propagating shock wave which is built up from a smoothly increasing input is exhibited in the various cases considered.

It is hoped that the present proposed method of solution will be employed in related problems whenever a complete solution is required.

A UNIAXIAL CONSTITUTIVE EQUATION FOR A COMPRESSIBLE THERMOELASTIC MATERIAL

In order to construct a uniaxial constitutive equation for a nonlinear thermoviscoelastic material we need first to derive the correspondent thermoelastic one. We shall adopt the quadratic material for which the internal energy is expanded as far as the second order terms of the strain components and the entropy S. Following Bland[11] notation the specific internal energy e is given for this material by

$$e(\gamma_{ij}, S) = T_0 S + \frac{\lambda}{2\rho} J_1^2 + \frac{\mu}{\rho} J_2 - K J_1 S + \frac{\eta}{2} S^2 \tag{1}$$

where λ, μ are the isentropic Lame' constants, T_0, ρ are the temperature and density in the undeformed state and K, η are material constants. In (1)

$$J_1 = \gamma_{ii}, \quad J_2 = \gamma_{ij}\gamma_{ij} \tag{2}$$

where γ_{ij} are the components of the Cauchy–Green strain tensor.

The temperature T is derived from e according to

$$T = \frac{\partial e}{\partial S} \tag{3}$$

so that for the quadratic material the following relation between the entropy and temperature is obtained

$$\eta S = T - T_0 + KJ_1. \tag{4}$$

In order to derive the required uniaxial stress-strain-temperature relation we impose the same three assumptions of Valanis and Sun [12] who constructed the uniaxial stress-strain relation for an incompressible material. These assumptions are: (a) Planes before deformation remain planes after deformation. (b) When deformation occurs, sections change size but not shape. (c) A change in size is gradual.

Let I_1, I_2, I_3 denote the principle invariants of the Cauchy–Green tensor

$$C_{kl} = \frac{\partial x_i}{\partial X_k}\frac{\partial x_i}{\partial X_l} \tag{5}$$

where x_k is the current configuration described by the original configuration X_k. The tensor C_{ij} is related to γ_{ij} according to

$$\gamma_{ij} = (C_{ij} - \delta_{ij})/2 \tag{6}$$

where δ_{ij} is the Kronecker delta. Then the previous three assumptions yield the following expressions for I_1, I_2, I_3

$$\left.\begin{aligned} I_1 &= \lambda_1^2 + \lambda_2^2 + \lambda_3^2 \\ I_2 &= \lambda_1^2\lambda_2^2 + \lambda_1^2\lambda_3^2 + \lambda_2^2\lambda_3^2 \\ I_3 &= \lambda_1^2\lambda_2^2\lambda_3^2 \end{aligned}\right\} \tag{7}$$

with $\lambda_2 = \lambda_3$,

$$\lambda_l = \frac{\partial x_i}{\partial X_l}\delta_{il} \quad (l \text{ is not summed}) \tag{8}$$

and X_1 is the position in the uniaxial direction whereas X_2, X_3 define the positions in the thickness directions.

J_1 and J_2 in (1) are related to I_1 and I_2 as follows

$$\begin{aligned} J_2 &= (I_1 - 3)/2 \\ J_2 &= (I_1^2 - 2I_1 - 2I_2 + 3)/4 \end{aligned} \tag{9}$$

The first Piola–Kirchhoff stress tensor is given by

$$L_{ij} = \rho \partial e / \partial \left(\frac{\partial u_i}{\partial X_j}\right) \tag{10}$$

where u_i are the displacement components.

In a uniaxial situation the principal stress components L_{22}, L_{33} must vanish. This yields after some lengthy manipulations the following relation between λ_2 and λ_1

$$\lambda_2^2 = \left[\frac{3\lambda + 2\mu}{2\rho} - 1{\cdot}5\frac{K^2}{\eta} + \frac{K}{\eta}(T - T_0) + 0{\cdot}5\left(\frac{K^2}{\eta} - \frac{\lambda}{\rho}\right)\lambda_1^2\right] \Big/ \left(\frac{\lambda + 2\mu}{\rho} - \frac{K^2}{\eta}\right). \tag{11}$$

The isothermal Lame' constants λ_I, μ_I are related to the isentropic ones by [13]

$$\left.\begin{aligned} \lambda_I &= \lambda - \rho K^2/\eta \\ \mu_I &= \mu \end{aligned}\right\}. \tag{12}$$

Furthermore, the coefficient of linear expansion is related to λ_I, μ_I, K, η, ρ by

$$\alpha = \frac{\rho K}{\eta(3\lambda_I + 2\mu_I)}. \tag{13}$$

Hence after some lengthy manipulations in which the dependence of L_{11} on λ_2 is eliminated using (11) we obtain the following expression for the uniaxial stress component

$$L_{11} \equiv \sigma = E_e(1 + 1{\cdot}5u_{,X} + 0{\cdot}5u_{,X}^2)u_{,X} - \alpha E_e(1 + u_{,X})(T - T_0) \tag{14}$$

where

$$E_e = \mu_I(3\lambda_I + 2\mu_I)/(\lambda_I + \mu_I)$$

is the Young modulus and $u_{,X} = (\partial u_1/\partial X_1)$

A UNIAXIAL CONSTITUTIVE EQUATION FOR A NONLINEAR THERMOVISCOELASTIC MATERIAL

A uniaxial stress-strain-temperature equation for a nonlinear thermoviscoelastic material is given by Schapery [4]. This equation, which is an extension of a linear constitutive equation to nonlinear behavior, is based on the theory of the thermodynamics of irreversible processes. By assuming a certain simple form for the free energy and entropy production, Schapery derived the following constitutive equation containing only single integral terms

$$Q_1 = \frac{\partial F_e}{\partial q_1} + a_F \int_{-\infty}^{t} [E(\psi - \psi') - E_e] \frac{\partial}{\partial t'} q_1 \, dt' - \alpha a_F \int_{-\infty}^{t} E(\psi - \psi') \frac{\partial}{\partial t'} \tilde{\theta} \, dt' \tag{15}$$

where q_1 is the single observed generalized coordinate chosen in this case as the displacement gradient $q_1 = u_{,X}$ so that Q_1 which is the single generalized force will be the uniaxial stress σ. F_e is the equilibrium free energy for which $T = T_0$, $E(\psi)$ is the relaxation function, $E_e = E(\infty)$ is the elastic modulus corresponding to the equilibrium state and

$$\tilde{\theta} = a_s(T - T_0)/a_F. \tag{16}$$

In (15), (16) $a_F(q_1)$, $a_s(q_1)$ are material functions which are equal to unity when $q_1 = 0$.

$\psi = \psi(t)$ is the strain-temperature reduced time which is related to the time t through

$$\psi(X, t) = \int_0^t \frac{dt}{a_{\epsilon T}} \tag{17}$$

with $a_{\epsilon T} = a_{\epsilon T}(q_1, T)$ being the thermomechanical shift factor and $\psi' = \psi(X, t')$.

For rubbery polymers, which can sustain very high strains before breaking, the shift factor is in general insensitive to the stress level whereas it depends very strongly on the temperature, see [1] and [5]. Also for those polymers the following relation exists[5]

$$\frac{\partial F_e}{\partial q_1} = E_e a_F q_1. \tag{18}$$

In the special case of a nonlinear thermoelastic material (15) reduces to

$$\sigma = \frac{\partial F_e}{\partial q_1} - \alpha E_e(T - T_0).$$

Comparing the above equation with (14) for the uniaxial constitutive equation of the quadratic material we have

$$\left.\begin{aligned} a_F &= 1 + 1{\cdot}5u_{,X} + 0{\cdot}5u_{,X}^2 \\ a_s &= 1 + u_{,X} \end{aligned}\right\}. \tag{19}$$

Accordingly, we obtain the following nonlinear uniaxial thermoviscoelastic constitutive equation

$$\begin{aligned} \sigma(X, t) = E_e a_F u_{,X} + a_F \int_{-\infty}^{t} [E(\psi - \psi') - E_e] \frac{\partial}{\partial t'} u_{,X}\, dt' \\ - a_F \alpha \int_{-\infty}^{t} E(\psi - \psi') \frac{\partial}{\partial t'} \tilde{\theta}\, dt' \end{aligned} \tag{20}$$

with the material functions a_F, a_s given by (19). The linear form of (20) is recovered when $a_F = a_s \equiv 1$.

In the isothermal case, equation (20) reduces to the uniaxial equation of Vogt and Schapery[14]. In reference [14] the problem of shock waves in viscoelastic rod was treated by employing the wave-front expansion method, and the complete solution was obtained numerically by Benveniste and Aboudi[15].

The relaxation function $E(\psi)$ is chosen in this paper in the form of exponential series

$$E(\psi) = E_1 + \sum_{r=2}^{N+1} E_r \exp(-\psi/t_r) \tag{21}$$

with $E(\psi)=0$ for $\psi<0$, $E_1=E_e$, E_r are constants and $t_r>0$ being relaxation constants. In the linear range (21) yields the generalized Maxwell model describing the viscoelastic behavior.

Substituting (21) in (20), the following relation is obtained

$$\sigma(X,t)=E_1a_Fu_{,X}+a_F\sum_{r=2}^{N+1}E_r\exp(-\psi/t_r)[\overset{(1)}{h}_r-\alpha\overset{(2)}{h}_r]-\alpha E_1a_s(T-T_0) \tag{22}$$

where

$$\overset{(1)}{h}_r(X,t)=\int_0^t\exp(\psi'/t_r)\frac{\partial}{\partial t'}u_{,X}\,dt' \tag{23}$$

and

$$\overset{(2)}{h}_r(X,t)=\int_0^t\exp(\psi'/t_r)\frac{\partial}{\partial t'}\tilde{\theta}\,dt'. \tag{24}$$

The uniaxial equation of motion can be written in the form

$$\begin{aligned}\rho u_{,tt}&=E_1[a_Fu_{,XX}+a_{F,X}u_{,X}]-\alpha E_1[a_sT_{,X}+a_{s,X}(T-T_0)]\\&+a_F\sum_{r=2}^{N+1}E_r\exp(-\psi/t_2)[\overset{(3)}{h}_r/t_r+\overset{(4)}{h}_r-\overset{(1)}{h}_r\psi_{,X}/t_r\\&-\alpha(\overset{(5)}{h}_r/t_r+\overset{(6)}{h}_r-\overset{(2)}{h}_r\psi_{,X}/t_r)]+a_{F,X}\sum_{r=2}^{N+1}E_r\exp(-\psi/t_r)[\overset{(1)}{h}_r-\alpha\overset{(2)}{h}_r]\end{aligned} \tag{25}$$

with

$$\overset{(3)}{h}_r(X,t)=\int_0^t\exp(\psi'/t_r)\psi'_{,X}\frac{\partial}{\partial t'}u_{,X}\,dt' \tag{26}$$

$$\overset{(4)}{h}_r(X,t)=\int_0^t\exp(\psi'/t_r)\frac{\partial}{\partial t'}u_{,XX}\,dt' \tag{27}$$

$$\overset{(5)}{h}_r(X,t)=\int_0^t\exp(\psi'/t_r)\psi'_{,X}\frac{\partial}{\partial t'}\tilde{\theta}\,dt' \tag{28}$$

$$\overset{(6)}{h}_r(X,t)=\int_0^t\exp(\psi'/t_r)\frac{\partial}{\partial t'}\tilde{\theta}_{,X}\,dt'. \tag{29}$$

THE ENERGY EQUATION FOR THE UNIAXIAL MOTION

The energy equation in terms of the generalized coordinates is given by [4]

$$\frac{\partial H}{\partial\psi}=\rho c_v\frac{a_F}{a_s}\frac{\partial\tilde{\theta}}{\partial\psi}+T_0a_s\sum_{i=1}^{N+1}\beta_i\frac{\partial q_i}{\partial\psi}-2a_F\tilde{D} \tag{30}$$

where H is the amount of heat gained by the material and c_v is the specific heat at constant volume. In (30) the q_i are the $N+1$ generalized coordinates and are composed in the present uniaxial case of one observed coordinate and N hidden coordinates $q_2,\ldots,q_{N+1}$. β_i are material constants which appear in the corresponding thermodynamic relations of the linear case and $\tilde{D}$ is the dissipation function given by

$$\tilde{D}=\frac{1}{2}\sum_{i,j=1}^{N}b_{ij}\frac{\partial q_i}{\partial\psi}\frac{\partial q_j}{\partial\psi} \tag{31}$$

with $b_{ij} = b_{ji}$ being material constants which like β_i appear in the thermodynamic linear relations.

In order to formulate the energy equation (30) and the associate dissipation function (31) in terms of the observed coordinate $q_1 = u_{,X}$ and temperature, we have to express the hidden coordinates in terms of the observed coordinates and temperature. Such a relation between the observed and hidden coordinates is given by Fung[16] for the linear isothermal viscoelastic case and is extended here to the present nonlinear thermoviscoelastic case. For this purpose let us employ the equation of evolution[4]

$$\sum_{j=1}^{N+1}\left[a_{ij}q_j + b_{ij}\frac{\partial q_i}{\partial \psi}\right] = \tilde{Q}_i + \beta_i\tilde{\theta}, \quad (i = 1, \ldots, N+1) \tag{32}$$

where $a_{ij} = a_{ji}$ are material constants like b_{ij} and β_i, and $a_{ij} = 0$ for $j = 2, \ldots, N+1$. In (32)

$$\tilde{Q}_1 = \left[Q_1 + a_F a_{11} q_1 - \frac{\partial F_e}{\partial q_1}\right]\Big/ a_F \tag{33}$$

with $\tilde{Q}_r = 0$ for $r = 2, \ldots, N+1$, and Q_1 is the observed generalized force which turns out to be the uniaxial stress σ for the present choice of $q_1 = u_{,X}$.

Following Fung[16] let us define B_{rs} $(r, s = 2, \ldots, N+1)$ to be a normalizing matrix such that

$$\sum_{l,s} B_{sr} b_{sl} B_{lk} = \delta_{rk} \qquad (k, l = 2, \ldots, N+1) \tag{34}$$

and

$$\sum_{l,s} B_{sr} a_{sl} B_{lk} = \mu_r \delta_{rk} \tag{35}$$

where μ_r are constants.

It follows after some lengthy manipulations which involve the application of Laplace transformation with respect to ψ to (32), employing (34) and (35) and inverting back, that

$$\frac{\partial q_r}{\partial \psi} = -\sum_s \left[P_s B_{rs}\frac{\partial q_1}{\partial \psi}\right] + \sum_s \left[B_{rs}(\omega_s R_s + \eta_s S_s)\right] \tag{36}$$

and

$$Q_1 = \frac{\partial F_e}{\partial q} + a_F \frac{\partial q_1}{\partial \psi}\left(b_{11} - \sum_s V_s\right) - \beta_1 a_F \tilde{\theta} + a_F \sum_s (\mu_s V_s R_s + M_s S_s) \tag{37}$$

where

$$P_r = \sum_s B_{sr} b_s, \quad b_r \equiv b_{1r} \tag{38}$$

$$\omega_r = \sum_s B_{sr} b_s \mu_r \tag{39}$$

$$\eta_r = \sum_s B_{sr} \beta_s \tag{40}$$

$$\left.\begin{aligned} R_r &= \exp(-\mu_r \psi)\, \overset{(1)}{h}_r \\ S_r &= \exp(-\mu_r \psi)\, \overset{(2)}{h}_r \end{aligned}\right\} \tag{41}$$

$$V_r = \sum_{l,s} b_s B_{sr} B_{lr} b_l \tag{42}$$

$$M_r = \sum_s b_r B_{rs} \eta_s. \tag{43}$$

Equation (36) is the required expressions for the hidden coordinates q_r in terms of the observed coordinate q_1 and temperature, whereas equation (37) yields a constitutive relation for the generalized force $Q_1 = \sigma$ in terms of the observed coordinate $q_1 = u_{,X}$ and temperature.

By comparing (37) with (15) and $E(\psi)$ given by (21) we readily obtain

$$b_{11} - \sum_s V_s = 0 \tag{44}$$

$$\beta_1 = \alpha E_1 \tag{45}$$

$$\mu_r V_r = E_r \tag{46}$$

$$\mu_r = 1/t_r \tag{47}$$

$$M_r = -\alpha E_r. \tag{48}$$

Moreover, these relations enable us also to express the dissipation function (31) in terms of the observed coordinate and temperature. Thus we can write after some expansions

$$\tilde{D} = \frac{1}{2} \sum_{k,l,r,s} b_{rs} B_{rk} B_{sl} [\omega_k \omega_l R_k R_l + \eta_k \eta_l S_k S_l + 2\omega_k \eta_l R_k S_l] \tag{49}$$

Employing (45), (46), (49) yields after some operations

$$\tilde{D} = \frac{1}{2} \sum_{r=2}^{N+1} E_r \mu_r [R_r - \alpha S_r]^2 \tag{50}$$

which is the required expression for the dissipation function.

As to the energy equation (30), we can express it in terms of the observed coordinate q_1 by separating the observed and hidden coordinates yielding

$$\frac{\partial H}{\partial \psi} = \rho c_v \frac{a_F}{a_s} \frac{\partial \tilde{\theta}}{\partial \psi} + T_0 a_s \beta_1 \frac{\partial q_1}{\partial \psi} + T_0 a_s \sum_{r=2}^{N+1} \beta_r \frac{\partial q_r}{\partial \psi} - 2a_F \tilde{D} \tag{51}$$

Using the expression for $\partial q_r / \partial \psi$ given by (36) and employing (45), (46) we obtain

$$\frac{\partial H}{\partial \psi} = \rho c_v \frac{a_F}{a_s} \frac{\partial \tilde{\theta}}{\partial \psi} + T_0 a_s E_1 \alpha \frac{\partial q_1}{\partial \psi} + \alpha T_0 a_s \sum_{r=2}^{N+1} E_r \left[\frac{\partial q_1}{\partial \psi} - \mu_r R_r + \alpha \mu_r S_r\right] - 2a_F \tilde{D}. \tag{52}$$

With this equation we incorporate the Fourier law

$$H = -kT_{,X} \tag{53}$$

where k is the heat conduction coefficient (assumed constant), yielding the final form

$$kT_{,XX} = \rho c_v \frac{a_F}{a_s}\tilde{\theta}_{,t} + \alpha T_0 a_s u_{,Xt}\sum_{i=1}^{N+1} E_i + \frac{a_s \alpha T_0}{a_{\epsilon T}}\sum_{r=2}^{N+1} E_r \exp(-\psi/t_r)[\alpha \overset{(2)}{h}_r - \overset{(1)}{h}_r]/t_r - 2a_F\tilde{D}/a_{\epsilon T} \quad (54)$$

and

$$\tilde{D} = \frac{1}{2}\sum_{r=2}^{N+1}\frac{E_r}{t_r}\exp(-2\psi/t_r)[\overset{(1)}{h}_r - \alpha \overset{(2)}{h}_r]^2. \quad (55)$$

The special case of a linear thermoviscoelastic material, whose viscoelastic properties are still temperature dependent, is obtained by setting $a_s = a_F = 1$ in the constitutive equation (22), the energy equation (54) and the dissipation function (55). Indeed, it can be easily checked that (22) reduces to the constitutive relation given by Hunter[8]. The energy equation in the linear case is given by Hunter but it is given in [8] in terms of the specific heat at constant pressure c_p rather than c_v as in the present formulation. By employing the relation between c_p and c_v for the uniaxial case, i.e.

$$c_p = c_v + \alpha^2 E(0)T_0/\rho \quad (56)$$

we can readily obtain from (54) with $a_F = a_s = 1$ the energy equation given by Hunter[8]. As to the dissipation function (55), it can be also transformed and shown to be identical with that given in [8].

We note that even with $a_F = a_s = 1$, the coupled "linear" thermoviscoelastic case in which the temperature is transient as well as nonuniform, there are two sources of nonlinearity: (1) The temperature dependence of the viscoelastic modulus through their explicit dependence on the reduced time $\psi(X, t)$ which depends on the unknown temperature field. (2) The appearance of the dissipation function which is quadratic in the displacement gradient, in the energy equation.

FINITE-DIFFERENCE FORMULATION

In this section we present a finite-difference method of solution to the equation of motion (25), the energy equation (54) and the associated dissipation function (55).

We start by transforming these equations to a nondimensional form by introducing the nondimensional variables for the length, time and temperature

$$\left.\begin{aligned} \xi &= \rho c_v c_0 X/k \\ \tau &= \rho c_v c_0^2 t/k \\ \theta &= (T - T_0)/T_0 \end{aligned}\right\} \quad (57)$$

where

$$c_0^2 = E(0)/\rho. \quad (58)$$

This yields the following form of the equation of motion

$$\begin{aligned} U_{,\tau\tau} = {} & G_1[A_F U_{,\xi\xi} + A_{F,\xi}U_{,\xi}] - \gamma G_1[A_s\theta_{,\xi} + A_{s,\xi}\theta] + A_F\sum_{r=2}^{N+1} G_r \exp(-\phi/\tau_r)[\overset{(3)}{H}_r/\tau_r + \overset{(4)}{H}_r - \overset{(1)}{H}_r\phi_{,\xi}/\tau \\ & - \gamma(\overset{(5)}{H}_r/\tau_r + \overset{(6)}{H}_r - \overset{(2)}{H}_r\phi_{,\xi}/\tau_r)] + A_{F,\xi}\sum_{r=2}^{N+1} G_r \exp(-\phi/\tau_r)[\overset{(1)}{H}_r - \gamma\overset{(2)}{H}_r] \end{aligned} \quad (59)$$

where

$$\gamma = \alpha T_0, \quad G_r = E_r/\rho c_0^2, \quad \tau_r = \rho c_v c_0^2 t_r/k,$$

$\phi(\xi, \tau)$ is the nondimensional reduced time

$$\phi(\xi, \tau) = \int_0^\tau \mathrm{d}\tau/\chi(U_{,\xi}, \theta)\,\mathrm{d}\tau \tag{60}$$

and χ is the nondimensional thermomechanical shift factor.

U is related to u as ξ to X in (57), A_F, A_s are the same expressions as a_F, a_s but with $u_{,X}$ replaced by $U_{,\xi}$ and $\overset{(l)}{H}_r(\xi, \tau)$ $(l = 1, \ldots, 6)$ are of the same form as $\overset{(l)}{h}_r(X, t)$ in (23–24), (26–29) but with $u, X, t, t', \psi, \psi', \tilde{\theta}$ replaced respectively with $U, \xi, \tau, \tau', \phi, \phi', \Theta$ where $\Theta = A_s\theta/A_F$. the energy equation (54) takes the form

$$\theta_{,\tau} = \theta_{,\xi\xi} - \frac{A_F}{A_s}\theta\left(\frac{A_s}{A_F}\right)_{,\tau} - \frac{\delta A_s}{\gamma} U_{,\xi\tau} + \frac{\delta A_s}{\gamma\chi}\sum_{r=2}^{N+1} G_r \exp(-\phi/\tau_r)[\overset{(1)}{H}_r - \gamma \overset{(2)}{H}_r]/\tau_r + \bar{D} \tag{61}$$

where

$$\bar{D} = \frac{\delta A_F}{\gamma^2\chi}\sum_{r=2}^{N+1} G_r \exp(-2\phi/\tau_r)[\overset{(1)}{H}_r - \gamma \overset{(2)}{H}_r]^2/\tau_r \tag{62}$$

and δ is the thermomechanical coupling constant

$$\delta = \alpha^2 T_0 E(0)/\rho c_v. \tag{63}$$

The numerical formulation is evaluated by introducing the spatial increment $\Delta\xi$ and temporal increment $\Delta\tau$, such that a function $f(\xi, \tau)$ is discretized in the form $f_i^n = f(i\Delta\xi, n\Delta\tau)$. The finite-difference scheme of the equation of motion (59) can be written in the form

$$U_i^{n+1} = 2U_i^n - U_i^{n-1} + (\Delta\tau)^2 L[U_i^n, \theta_i^n] \tag{64}$$

where $n = 2, 3, \ldots$ and for the region $\xi \geq 0$, $i = 0, 1, \ldots$, and L is a spatial difference operator which is obtained by discretizing the right hand side of (59) and approximating $U_{,\xi}$, $U_{,\xi\xi}$, θ_ξ with their corresponding central difference versions which are correct up to second order in $\Delta\xi$. Also, L will involve the expressions θ_i^n, ϕ_i^n, $(\phi_{,\xi})_i^n$ and $\overset{(l)}{H}{}_{ri}^n$ $(r = 2, \ldots, N+1$ and $l = 1, \ldots, 6)$ and the computation of these expressions at every time step will be explained in the sequel.

According to (64) it is possible to compute step by step the displacements U_i^{n+1} at the time level $\tau + \Delta\tau$ when all other quantities are known at times τ and $\tau - \Delta\tau$ for all $i = 0, 1, 2, \ldots$.

For the energy equation (61) an implicit finite-difference scheme will be employed in order to prevent the need of considerably small time increments $\Delta\tau$ known to be necessary when approximating a linear parabolic heat equation by a simple explicit method. We choose as in [10] the Crank–Nicolson implicit scheme which yields for (61) the following system of algebraic equations at every time step

$$-\epsilon\theta_{i-1}^{n+1} + (2\epsilon + 1)\theta_i^{n+1} - \epsilon\theta_{i+1}^{n+1} = \epsilon(\theta_{i+1}^n + \theta_{i-1}^n) + (1 - 2\epsilon)\theta_i^n + \Delta\tau p_i^n \tag{65}$$

where

$$p_i^n = \frac{0\cdot 5\theta_i^n}{1+0\cdot 5\bar{U}_i^n}\bar{\bar{U}}_i^n - \frac{\delta}{\gamma}\bar{A}_s\bar{\bar{U}}_i^n + \frac{\delta}{\gamma}\frac{\bar{A}_s}{\chi_i^n}\sum_{r=2}^{N+1} G_r \exp(-\phi_i^n/\tau_r)[\overset{(1)}{H}{}_{ri}^n - \gamma\overset{(2)}{H}{}_{ri}^n]/\tau_r + \bar{D}_i^n \tag{66}$$

$$\bar{D}_i^n = \frac{\delta}{\gamma^2}\frac{\bar{A}_F}{\chi_i^n}\sum_{r=2}^{N+1} G_r \exp(-2\phi_i^n/\tau_r)[\overset{(1)}{H}{}_{ri}^n - \gamma\overset{(2)}{H}{}_{ri}^n]^2/\tau_r \tag{67}$$

and

$$\epsilon = 0\cdot 5\,\Delta\tau/(\Delta\xi)^2.$$

In (65–67) $\bar{U}_i^n$, $\bar{\bar{U}}_i^n$ are the central difference approximations of $U_{,\xi}$ $U_{,\xi\tau}$ respectively and $\bar{A}_F$, $\bar{A}_s$ are the same expressions in $\bar{U}_i^n$ as A_F and A_s in terms of $U_{,\xi}$.

Equation (66) consists a system of M algebraic equations in the unknowns θ_i^{n+1} ($i = 1,\ldots,M$) where $M = \xi_1/\Delta\xi$ with ξ_1 being a point at which the values of the displacements and temperatures have no influence for a preassigned degree of accuracy on their values at the range $0<\xi<\xi_1$ for a given range of space and time.

The quantities ϕ_i^{n+1}, $(\phi_{,\xi})_i^{n+1}$, $\overset{(l)}{H}{}_{ri}^{n+1}$ are computed at every time step by employing the trapezoidal integration rule. For example

$$\overset{(1)}{H}{}_{ri}^{n+1} \approx \overset{(1)}{H}{}_{ri}^n + 0\cdot 5[\exp(\phi_i^{n+1}/\tau_r) + \exp(\phi_i^n/\tau_r)]\cdot(\bar{U}_i^{n+1} - \bar{U}_i^n). \tag{68}$$

The previous schemes are associated with initial conditions and boundary conditions (at $\xi = 0$). These schemes were found satisfactory and accurate whenever they applied to problems possessing a continuous solution. On the other hand when they are applied to a problem in which discontinuities are encountered, such as shock waves, numerical oscillations which are quite strong are formed which may in time distort the true solution, see also [9] for more detailed discussion.

In order to remove these oscillations an iteration process, previously employed in [9] and [10], is generalized here and applied to the explicit scheme (64) in the form

$$\begin{aligned} U_i^{n+1,m} = 2U_i^n - U_i^{n-1} + (\Delta\tau)^2\{w_3 L[U_i^{n+1,m-1}, \theta_i^{n+1,m-1}] + w_2 L[U_i^n, \theta_i^n] \\ + w_1 L[U_i^{n-1}, \theta_i^{n-1}]\}/(w_1 + w_2 + w_3) \end{aligned} \tag{69}$$

where m is the number of the iteration $m = 1, 2, \ldots$, w_i are weight numbers and $U_i^{n+1,0}$, $\theta_i^{n+1,0}$ are defined to be equal to U_i^{n+1}, θ_i^{n+1} given by (64), (65) respectively. The quantities $\theta_i^{n+1,m}$, $\phi_i^{n+1,m}$, $(\phi_{,\xi})_i^{n+1,m}$, $\overset{(l)}{H}{}_{ri}^{n+1,m}$ are computed by using (65), (68), etc. whenever $U_i^{n+1,m}$ have been calculated by (69) so that everything is known for the next iteration $m+1$.

Actual computations with (69) show that one iteration only ($m = 1$) removes almost the oscillations near the shock. Accordingly, all the results given in this paper are obtained by applying one iteration only.

For a nonlinear thermoelastic half-space the present numerical schemes reduce basically to those described in [10]. In [10] the numerical results were shown to yield accurate results by comparison with several analytic conclusions which could be drawn under special boundary conditions.

RESULTS

Consider a thermoviscoelastic semi-infinite rod $0 \le \xi < \infty$ subjected to the following boundary conditions at $\xi = 0$

(a) Mechanical loading

$$\left.\begin{aligned} U_{,\xi} &= e_0 f(\tau) \\ \theta &= 0 \end{aligned}\right\} \text{ at } \xi = 0 \tag{70}$$

(b) Thermal loading

$$\left.\begin{aligned} U_{,\xi} &= 0 \\ \theta &= \theta_0 f(\tau) \end{aligned}\right\} \text{ at } \xi = 0 \tag{71}$$

where e_0, θ_0 are constants and $f(\tau)$ is a smooth input function which rises from zero at $\tau = 0$ up to 1 at $\tau = 2\Delta$, such that 2Δ determines the rise time of the input (see [9]). The rod is assumed to be initially at rest prior to the application of the loading.

The following cases will be considered:

(1) Nonlinear elastic rod

$$G_1 = 1, \quad G_r = 0 \quad (r \ge 2).$$

(2) Nonlinear viscoelastic rod

$$G_1 = G_2 = \tfrac{1}{2}, \quad G_r = 0 \quad (r \ge 3), \quad \tau_2 = 1.$$

(3) Nonlinear thermoelastic rod

$$G_1 = 1, \quad G_r = 0 \quad (r \ge 2).$$

(4) Nonlinear thermoviscoelastic rod with temperature independent properties

$$G_1 = G_2 = \tfrac{1}{2}, \quad G_r = 0 \quad (r \ge 3), \quad \tau_2 = 1, \quad \chi \equiv 1.$$

(5) Nonlinear thermoviscoelastic rod with temperature dependent properties. The coefficients in this case are as given in case (4) but with the shift function

$$\chi(T) = \exp\left[-9(T - T_0)/(100 + T - T_0)\right], \quad T_0 \approx 273°\mathrm{K}.$$

This is the well known so called WLF shift function which applied for wide variety of polymers above their glass transition temperature for which the rubberlike behavior is observed.

The results given herein were obtained with the spatial increment $\Delta\xi = 0{\cdot}01$, rise time input $2\Delta = 0{\cdot}2$, thermomechanical coupling $\delta = 0{\cdot}1$ and $\gamma = 0{\cdot}1$. For every case the corresponding linear response can be obtained by setting $a_F = a_s \equiv 1$.

(a) *Mechanical loading*

In Fig. 1 the displacement gradients $U_{,\xi}$ and the temperature θ versus the time τ are shown for the boundary conditions (70) of mechanical loading with $e_0 = -0{\cdot}25$ at station $\xi = 0{\cdot}3$ for three cases of nonlinearly elastic, viscoelastic and thermoelastic rod. In the elastic case it can be

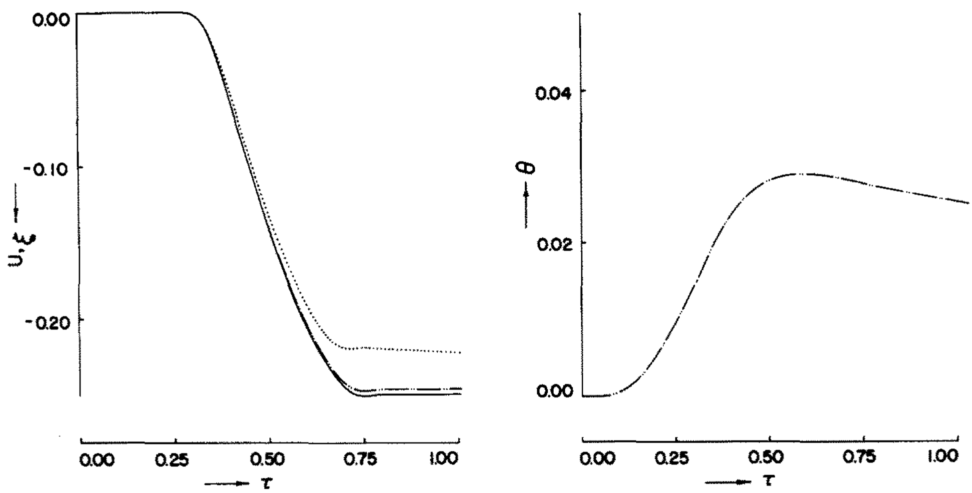

Fig. 1. Mechanical negative loading. Plot of the displacement gradients and temperature versus time at station $\xi = 0{\cdot}3$, for a nonlinearly elastic (———), viscoelastic (· · · · · ·) and thermoelastic (—·—·—) rod.

shown[17] that this type of a smooth monotonically decreasing from zero loading yields for the present material a simple wave solution given by

$$U_{,\xi}(\xi, t) = e_0 f(\tau - \xi/c), \quad e_0 < 0 \tag{72}$$

where $c(U_{,\xi})$ is the velocity of the acceleration waves given by

$$c(U_{,\xi}) = [1{\cdot}5U_{,\xi}^2 + 3U_{,\xi} + 1]^{1/2} \tag{73}$$

subject to the condition that $U_{,\xi} > -1$ and the hyperbolicity condition $c > 0$. As in [9] the simple wave solution (72) provides an analytical check to the numerical scheme, and in Fig. 1 the analytical and numerical solutions are up to the scale of the plot indistinguishable.

The presence of the separate viscoelastic and thermoelastic effects can be well observed in the Fig. 1 by comparing the elastic solution (72) with the pertinent curves, showing clearly the distorting effects on the simple wave solution caused by those mechanisms. The dissipation, attenuation and the excited temperature due to the thermomechanical coupling are fairly weak for the present choice of the coupling coefficient.

In Fig. 2 the combined thermal and viscoelastic effects are shown for a thermoviscoelastic rod whose viscoelastic properties are temperature dependent as against the case of thermal independent properties. Here too the excited temperature field is fairly small. Nevertheless, the effect of this dependence on the temperature is well observed especially in the plot of the displacement gradient due to the strong dependence of the shift factor on the temperature.

The linear thermoviscoelastic case with temperature dependent properties is also reproduced exhibiting a remarkable deviation from the nonlinear one. Indeed it is clearly seen that the displacement gradient and temperature are larger and less gradual in the linear case. This spreading of the curves is not unexpected since nonlinearity causes the shape of the present type of loading to spread as the disturbance propagates within the medium.

A completely different type of effects are obtained when applying the mechanical loading (70)

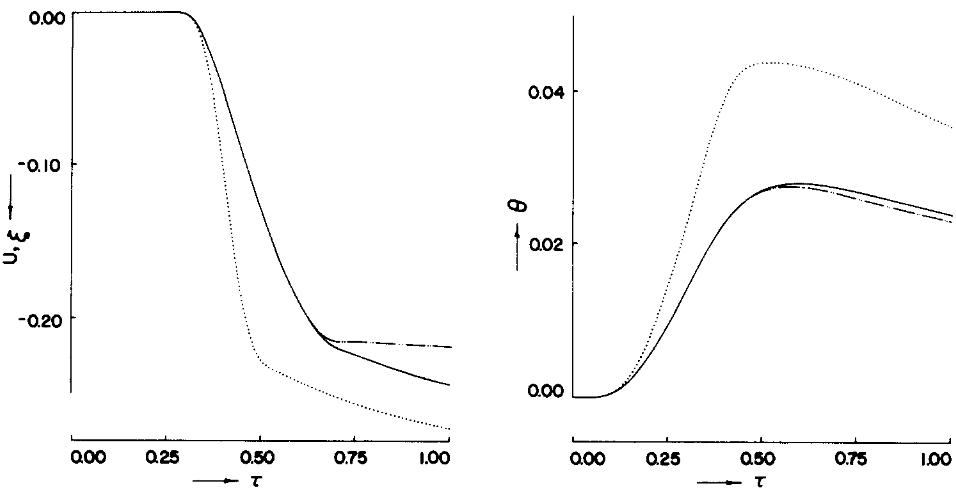

Fig. 2. Mechanical negative loading. Plot of the displacement gradients and temperatures versus time at station $\xi = 0{\cdot}3$, for a nonlinearly thermoviscoelastic rod with temperature dependent properties (———), temperature independent properties (–·–·–) and for a linearly thermoviscoelastic rod with temperature dependent properties (······).

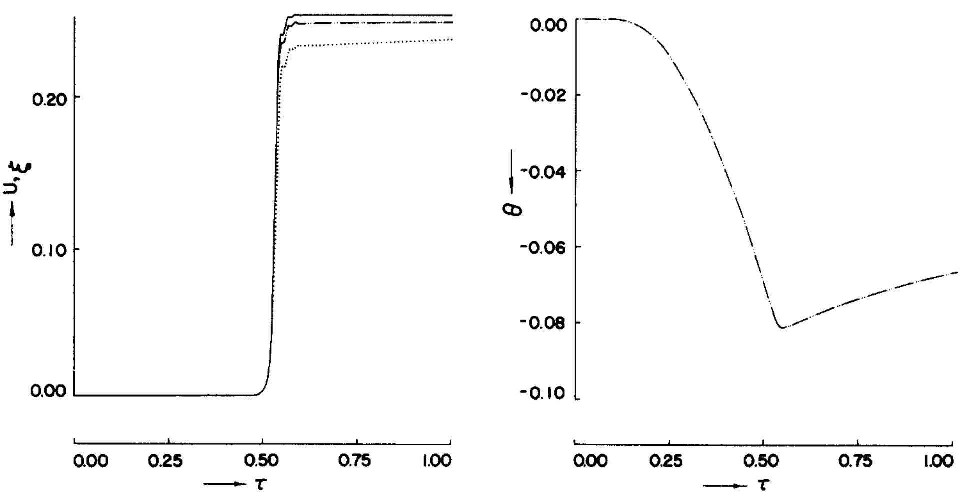

Fig. 3. Same as Fig. 1 but for a mechanical positive loading and at station $\xi = 0{\cdot}5$.

but with $e_0 > 0$. In this case the smoothly rising input applied on the boundary builts up into a propagating shock within the material. In Fig. 3 the displacement gradients and temperature in the nonlinearly elastic, viscoelastic and thermoelastic cases are shown for $e_0 = 0{\cdot}25$ at station $\xi = 0{\cdot}5$ where the shock is present.

For the nonlinear elastic rod, when the medium is previously undeformed, the velocity of the shock is given by

$$V = [A_F(U^*_{,\xi})]^{1/2} \tag{74}$$

where $U_{,\xi}^{*}$ is the jump in $U_{,\xi}$ across the shock, and it is in excellent agreement with the arrival time of the shock obtained by the numerical solution.

In the viscoelastic case the shock wave is attenuated, whereas the thermoelastic case yields a less attenuated shock wave (for the present choice of the coupling constant). Note that in the latter case the temperature variation is continuous so that the shock wave is isothermal as it can be verified[11] for the present choice of the heat flux equation (53).

In Fig. 4 the combined viscoelastic and thermal effects are shown fort the same type of loading at the same station. Here too a comparison between the cases of temperature dependent and independent properties is exhibited showing an increasing in attenuation in the first case as

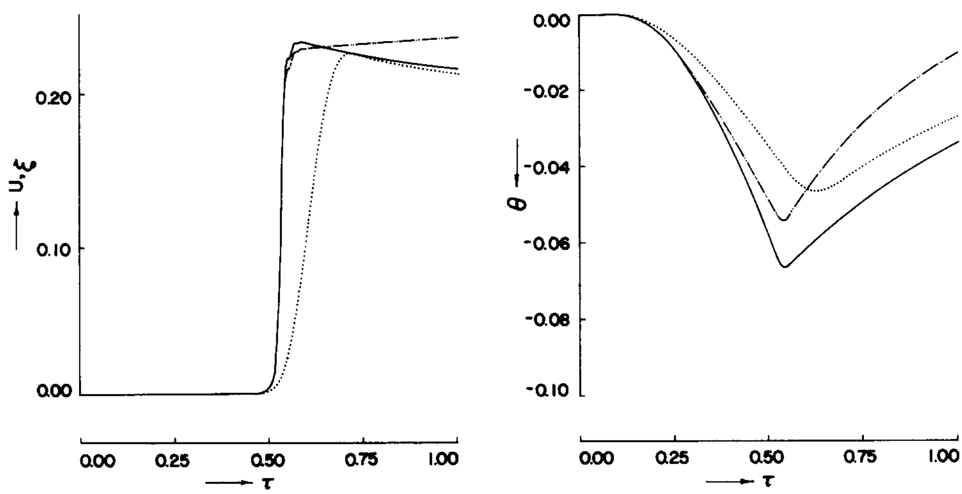

Fig. 4. Same as Fig. 2 but for a mechanical positive loading and at station $\xi = 0{\cdot}5$.

well as different behavior in both cases for larger times after the arrival of the shock. The response of a linear thermoviscoelastic rod (with temperature dependent properties) is also shown exhibiting very clearly the absence of the shock and yielding instead a smooth solution. Thus the general feature is preserved as in the purely elastic solution: for a monotonically input decreasing from zero, the nonlinearity causes a spreading to the propagating disturbance within the material as compared to the linear response, whereas for a monotonically increasing input from zero, the nonlinearity causes shock formation at a certain location and a shock wave propagates within the material as contrasted to the smooth propagating disturbance in the linear case.

(b) *Thermal loading*

In Fig. 5 the displacement gradient and temperatures are shown for the thermal loading conditions (71) with $\theta_0 = 0{\cdot}1$ at station $\xi = 0{\cdot}3$ within the rod. In the present case due to the value of the thermomechanical coupling constant chosen, relatively low values of displacement gradients are obtained and the main effect of the temperature field is to serve as an intrinsic time scale of the material at which the viscoelastic modulus is evaluated. This effect is exhibited in Fig. 5 by the two curves which show the influence of the temperature dependent and independent properties on the displacement gradient. It is well seen that this dependence on the relatively high positive temperature variations cause severe attenuation to the propagating mechanical

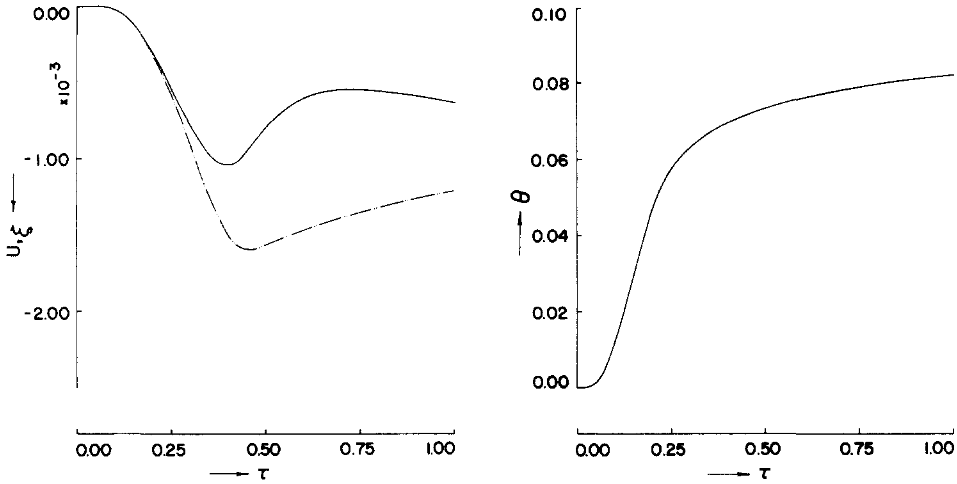

Fig. 5. Thermal loading. Plot of the displacement gradients and temperatures versus time at station $\xi = 0{\cdot}3$, for a nonlinearly thermoviscoelastic rod with temperature dependent properties (——) and temperature independent properties (–·–·–).

disturbance. The temperature variations on the other hand coincide in both cases due to the relatively small excited mechanical disturbances.

By applying the same thermal loading (71) but with $\theta_0 = -0{\cdot}1$ (not shown in the Figure) we obtain that the temperature dependent properties case yields higher values for the mechanical disturbances as compared with the temperature independent case. This is expected since the negative values of the temperature field in the present situation produce an intrinsic time scale such that the temperature dependent uniaxial modulus is evaluated near its initial value at which it attains its maximum.

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